

GEOMETRIC STRUCTURES ON LOOP AND PATH SPACES

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ABSTRACT. It is known that the loop space associated to a Riemannian manifold admits a quasi-symplectic structure. This article shows that this structure is not likely to recover the underlying Riemannian metric by proving a result that is a strong indication of the “almost” independence of the quasi-symplectic structure with respect to the metric. Finally conditions to have contact structures on these spaces are studied.

1. INTRODUCTION

The loop space $\mathcal{L}(M)$ of a manifold M comes equipped with a natural section of its tangent bundle defined as

$$\begin{aligned}\alpha : \mathcal{L}(M) &\rightarrow T\mathcal{L}(M) \\ \gamma &\rightarrow \gamma'.\end{aligned}$$

Whenever we fix a Riemannian metric g on M we can define an associated metric on the space of loops as

$$(g_{\mathcal{L}})_{\gamma}(X, Y) = \int_0^1 g(X(t), Y(t)) dt,$$

where $X, Y \in T_{\gamma}\mathcal{L}(M) \simeq \Gamma(S^1, \gamma^*TM)$ are two tangent vectors. This metric gives us an isomorphism between $T\mathcal{L}(M)$ and $T^*\mathcal{L}(M)$. Therefore α and $g_{\mathcal{L}}$ allow us to define a 1-form

$$(1) \quad \mu(X) = \frac{1}{2} \int_0^1 g(X(t), \gamma'(t)) dt,$$

whose exterior differential $\omega = d\mu$ happens to be quasi-symplectic. This means that the kernel of the form is finite dimensional, specifically

$$\ker(\omega)_{\gamma} = \{X \in \Gamma(S^1, \gamma^*TM) ; \nabla_{\gamma'} X = 0\}.$$

(We assume throughout this article that the spaces considered are in the C^{∞} category and have a natural Fréchet structure, unless something else is declared.)

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This quasi-symplectic structure can be enriched in many cases. This is well known in the case of loop groups (i.e., M is a Lie group). In this case it is possible to define an integrable complex structure making a finite codimensional closed manifold of a loop group into a Kähler manifold.

Extending our space to the path space defined as

$$\mathcal{P}(M) = \{\gamma : [0, 1] \rightarrow M\},$$

we still have the canonical section of the tangent bundle given by

$$\begin{aligned} \alpha : \mathcal{P}(M) &\rightarrow T\mathcal{P}(M) \\ \gamma &\rightarrow \gamma'. \end{aligned}$$

As in the loop space we will easily check that equation (1) is a 1-form whose differential is symplectic.

Proposition 1.1. *The 2-form $\omega = d\mu$ in $\mathcal{P}(M)$ induces a symplectic structure. Moreover $(\mathcal{L}(M), \omega)$ is a closed quasi-symplectic submanifold of $\mathcal{P}(M)$.*

It could be thought that the symplectic structure makes life easier in comparison to the quasi-symplectic one. At least, in terms of the stability of the structure this is not the case. In particular, we prove

Theorem 1.2. *Fix a smooth closed manifold M of even dimension. Denote as ω_g the quasi-symplectic form on $\mathcal{L}(M)$ associated to a Riemannian metric g on M . Then given two Riemannian metrics g_0 and g_1 on M , there exists a smooth isotopy ϕ_ϵ on $\mathcal{L}(M)$ such that $(\phi_\epsilon)_*\omega_{g_0}$ is ϵ -close to ω_{g_1} in L^2 -norm.*

The proof of this result cannot be generalized to the case of $\mathcal{P}(M)$. Also, the theorem cannot be improved to obtain an isotopy that matches ω_{g_0} and ω_{g_1} on the nose. Certainly, allowing ϵ go to zero makes the norm of the isotopy go to infinity and so discontinuities are developed. So the result can be understood as a sort of “approximate uniqueness” of the quasi-symplectic structure.

The proof of Theorem 1.2 is based on an adaptation of Moser’s trick to this setting. It is surprising that this type of argument works in an infinite-dimensional non-compact setting, since Moser’s trick needs the compactness of the manifold. Somehow, the compactness of the underlying manifold makes the job in our case.

This shows that the symplectic geometry of the loop space probably does not recover the Riemannian geometry of the underlying manifold in the even dimensional case. This statement will be clearer after the proof of Theorem 1.2, in which the geometric obstruction for the complete “uniqueness” of the quasi-symplectic structure is shown.

Finally we discuss how to find contact hypersurfaces in loop (and path) spaces. The most natural construction is given by

Theorem 1.3. *Assume that the Riemannian manifold (M, g) admits a vector field X which satisfies $L_X g = g$ and is locally gradient-like, then the lift*

of X to $\mathcal{L}(M)$ (respectively $\mathcal{P}(M)$) is a Liouville vector field for the length function.

This shows in particular that stabilizing the manifold M , i.e. considering $M \times \mathbb{R}$, we obtain contact hypersurfaces in the loop space.

2. SYMPLECTIC STRUCTURE

2.1. Basic definitions. The quasi-symplectic structure in the space of loops of a Riemannian manifold is defined by taking the exterior differential of the 1-form μ given by equation (1). To do that we recall the formula

$$(2) \quad d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]),$$

which is valid for any 1-form α and it does not depend on the vector fields X, Y chosen to extend $X(\gamma)$ and $Y(\gamma)$ for a given point (loop) $\gamma \in \mathcal{L}(M)$. In our case we start with two vectors $U, V \in \Gamma(S^1, \gamma^*TM) \simeq T_\gamma \mathcal{L}(M)$. First define

$$\theta : (-\varepsilon, \varepsilon)^2 \times S^1 \rightarrow M,$$

satisfying:

- (i) $\theta(0, 0, t) = \gamma(t)$,
- (ii) $\frac{\partial \theta}{\partial u}(0, 0, t) = U(t)$,
- (iii) $\frac{\partial \theta}{\partial v}(0, 0, t) = V(t)$.

And so define $\gamma' = \frac{\partial \theta}{\partial t}$, $\hat{U} = \frac{\partial \theta}{\partial u}$ and $\hat{V} = \frac{\partial \theta}{\partial v}$. They clearly satisfy

$$(3) \quad [\hat{U}, \hat{V}] = 0,$$

since they are derivatives of the coordinates of a parametrization. This allows us to compute

$$(4) \quad \begin{aligned} \hat{U}(\mu(\hat{V})) &= \frac{d}{du} \left(\frac{1}{2} \int_0^1 g_{\theta(u, 0, t)} \left(\frac{\partial \theta}{\partial v}(u, 0, t), \frac{\partial \theta}{\partial t}(u, 0, t) \right) dt \right) \\ &= \frac{1}{2} \int_0^1 (g_{\theta(0, 0, t)}(\nabla_{\hat{U}} \hat{V}, \gamma'(t)) + g_{\theta(0, 0, t)}(V, \nabla_{\hat{U}} \gamma')) dt. \end{aligned}$$

In the same way, we obtain

$$\hat{V}(\mu(\hat{U})) = \frac{1}{2} \int_0^1 (g_{\theta(0, 0, t)}(\nabla_{\hat{V}} \hat{U}, \gamma'(t)) + g_{\theta(0, 0, t)}(U, \nabla_{\hat{V}} \gamma')) dt.$$

We are using the torsion-free Levi-Civita connection for the computations, so $\nabla_{\gamma'} \hat{U} = \nabla_{\hat{U}} \gamma'$ and $\nabla_{\gamma'} \hat{V} = \nabla_{\hat{V}} \gamma'$. Also $\nabla_{\hat{U}} \hat{V} - \nabla_{\hat{V}} \hat{U} = [\hat{U}, \hat{V}] = 0$. We shall use the notation $\nabla_{\gamma'} U = \frac{\partial U}{\partial t}$. So we have, by applying the formula (2), that

$$(5) \quad \begin{aligned} \omega(U, V) &= \omega(\hat{U}, \hat{V}) = d\mu(\hat{U}, \hat{V}) = \hat{U}(\mu(\hat{V})) - \hat{V}(\mu(\hat{U})) = \\ &= \frac{1}{2} \int_0^1 \left(g_{\gamma(t)} \left(V, \frac{\partial U}{\partial t} \right) - g_{\gamma(t)} \left(U, \frac{\partial V}{\partial t} \right) \right) dt. \end{aligned}$$

Moreover we have

$$(6) \quad 0 = \int_0^1 \left(\frac{d}{dt} g(U, V) \right) dt = \int_0^1 \left(g\left(\frac{\partial U}{\partial t}, V\right) + g\left(U, \frac{\partial V}{\partial t}\right) \right) dt,$$

which implies

$$(7) \quad \omega(U, V) = \int_0^1 g\left(\frac{\partial U}{\partial t}, V\right) dt.$$

Now the kernel of this 2-form at a point γ is given by the parallel vector fields along γ . Therefore $\dim \ker(\gamma) \leq n$. There are several ways of removing the kernel. The simplest one is to fix a point $p \in M$ and to define

$$\mathcal{L}_p(M) = \{\gamma \in \mathcal{L}(M) ; \gamma(0) = p\}.$$

This forces the tangent vectors to satisfy

$$X \in T_\gamma \mathcal{L}_p(M) \Rightarrow X \in \Gamma(S^1, \gamma^* TM), \quad X(0) = 0.$$

Therefore any parallel vector field is null. So the manifold $\mathcal{L}_p(M)$ is symplectic.

Extend our space to $\mathcal{P}(M)$ where it is still possible to repeat all the previous computations. We highlight the differences. The equation (4) is exactly the same as it is symmetric. The equation (5) remains also without changes. We just need to rewrite equation (6) which is not true anymore and so the final expression for the exterior differential of μ becomes

$$\begin{aligned} d\mu(U, V) &= \omega(U, V) = \int_0^1 \left(g\left(\frac{\partial U}{\partial t}, V\right) - \frac{1}{2} \frac{d}{dt} g(U, V) \right) dt \\ &= \int_0^1 g\left(\frac{\partial U}{\partial t}, V\right) dt - \frac{g(U(1), V(1)) - g(U(0), V(0))}{2}. \end{aligned}$$

Is is obviously a closed (being exact) form. Let us compute its kernel. Assume that $X \in \ker(\omega)_\gamma$. Considering $\omega(X, V) = 0$ for all vectors $V \in T_\gamma \mathcal{P}(M)$ with $V(0) = V(1) = 0$, we obtain that

$$\frac{\partial X}{\partial t} = 0.$$

Now by choosing all $V \in T_\gamma \mathcal{P}(M)$ with $V(0) \neq 0$ and $V(1) = 0$, we conclude that $X(0) = 0$. By parallel transport, $X = 0$ and so the kernel of ω is trivial. Hence this form is symplectic. This proves Proposition 1.1.

2.2. Almost complex structures. There is a canonical almost-complex structure compatible with ω in $\mathcal{L}_p M$. Let us construct it. Given a curve $\gamma : [0, 1] \rightarrow M$, denote P_s^t the parallel transport isometry along γ . There is an isometric isomorphism between $\gamma^* TM$ and the trivial $T_{\gamma(0)} M$ bundle over I with constant metric $g_{\gamma(0)}$. This allows to translate any section $U(t) \in \gamma^* TM$ to a section $P_t^0(U(t)) = \hat{U}(t) \in T_{\gamma(0)} M$. This gives rise to a “développement” map

$$T_\gamma \mathcal{L}_p \cong \mathcal{L}_0(T_{\gamma(0)} M).$$

Note that if we apply this map to $\gamma'(t)$ itself, we get $x(t) \in T_{\gamma(0)}M$. Now we define

$$a(t) = \int_0^t x(s)ds,$$

which is known as the “développement de Cartan” of the curve γ in the tangent space $T_{\gamma(0)}M$. As the covariant derivative along γ becomes the ordinary derivative in $T_{\gamma(0)}M$, we have that γ is a geodesic just when its développement de Cartan is a line.

Define the almost complex structure \hat{J} in $T_\gamma \mathcal{L}_p(M)$ as follows: take any vector field $U \in \Gamma(S^1, \gamma^* TM)$ and compute its “développement” \hat{U} . Recall that $\hat{U}(0) = \hat{U}(1) = 0$ since $U(t) \in T_\gamma \mathcal{L}_p(M)$. Fixing an isomorphism $T_{\gamma(0)}M \cong \mathbb{R}^n$, we have $\hat{U}(t) \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$. Take its Fourier series expansion,

$$\hat{U}(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i k t},$$

where $a_k \in \mathbb{C}^n$ and $a_{-k} = \bar{a}_k$. Then define

$$(8) \quad \tilde{J}(\hat{U})(t) = \sum_{k<0} (-ia_k) e^{2\pi i k t} + a_0 + \sum_{k>0} ia_k e^{2\pi i k t},$$

We subtract the constant vector $J(\hat{U})(0)$ to get the almost-complex structure. So we have

$$\hat{J}(\hat{U})(t) = \tilde{J}(\hat{U})(t) - \tilde{J}(\hat{U})(0) \in T_\gamma \mathcal{L}_p(M).$$

To check that it is an almost complex structure we compute

$$\begin{aligned} \hat{J}\hat{J}(\hat{U}) &= \hat{J}(\tilde{J}(\hat{U}) - \tilde{J}(\hat{U})(0)) = \\ &= \tilde{J}\tilde{J}(\hat{U}) - \tilde{J}(\hat{U})(0) - \tilde{J}\tilde{J}(\hat{U})(0) + \tilde{J}(\hat{U})(0) = \\ &= \tilde{J}\tilde{J}(\hat{U}) - \tilde{J}\tilde{J}(\hat{U})(0) = \\ &= (-\hat{U} + 2a_0) - (-\hat{U}(0) + 2a_0) = -\hat{U}. \end{aligned}$$

So we obtain an almost complex structure on $\mathcal{L}_0(T_{\gamma(0)}M)$, then by using the “développement” we have an almost complex structure in $T_\gamma \mathcal{L}_p(M)$, that is, in $\mathcal{L}_p(M)$.

To check that \hat{J} is compatible with the symplectic form ω we just compute the value of ω when trivialized in the “développement”, to obtain

$$(9) \quad \omega(\hat{U}, \hat{V}) = \omega \left(\sum_p a_p e^{2\pi i p t}, \sum_q b_q e^{2\pi i q t} \right) = \sum_k 2\pi i k \operatorname{Re} \langle a_k, b_k \rangle,$$

where $a_k, b_k \in \mathbb{C}^n$, and \langle, \rangle is the standard Hermitian product in \mathbb{C}^n . The associated metric

$$g(\hat{U}, \hat{V}) = \omega(\hat{U}, J\hat{V}) = \sum_{k>0} 2\pi k \operatorname{Re}(\langle a_k, b_k \rangle + \langle a_{-k}, b_{-k} \rangle)$$

is clearly Riemannian. Moreover, \hat{J} is smooth. To check it, recall that the map

$$\begin{aligned} \mathcal{F} : \{f \in \mathcal{C}^\infty(S^1, \mathbb{R}^n) ; f(0) = 0\} &\rightarrow SS \subset (\mathbb{C}^n)^\infty \\ f &\mapsto (a_1, a_2, a_3, \dots), \end{aligned}$$

where SS is the Schwartz space of sequences of vectors in \mathbb{C}^n with decay faster than polynomial, and $\{a_k\}$ are the Fourier coefficients of f , is a topological isomorphism (we take in SS the Fréchet structure given by the norms $\|(a_k)\|_t = \sum k^t |a_k|$). (Note that the Fourier coefficients satisfy $a_{-k} = \bar{a}_k$ and $a_0 = -\sum_{k \neq 0} a_k$.) The map \hat{J} is conjugated under \mathcal{F} to the map

$$\begin{aligned} \mathcal{J} : SS &\rightarrow SS, \\ (a_1, a_2, \dots) &\mapsto (ia_1, ia_2, \dots), \end{aligned}$$

which is smooth (actually an isometry). So \hat{J} is smooth. This corrects the folklore statement saying that this almost complex structure is not smooth in general (see [Wu95, pag. 355]).

In the case in which M is a Lie group, there is an alternative way of defining an almost complex structure for the space of loops based at the neutral element $e \in G$. To do it we just use the left multiplication to take the tangent space $T_\gamma \mathcal{L}_e(G)$ to $\Gamma(S^1, T_e G)$, so we obtain an isomorphism $T_\gamma \mathcal{L}_e(G) \simeq \Gamma(S^1, \mathbb{R}^n)/\mathbb{R}^n$, preserving the metric by construction (the quotient is by the constant maps). So every particular vector field $X \in T_\gamma \mathcal{L}_e(G)$ is transformed via the isomorphism to a loop in \mathbb{R}^n . Recall that the isomorphism does not coincide with the one induced by the “développement” unless the group is flat (an abelian group). Once we have set up the previous identification, the formula (8) provides again an almost complex structure. Again we remark that it does not coincide with the previous one in the cases when both are defined.

2.3. Uniqueness. One may ask how canonical the symplectic structures on the loop spaces are. Such symplectic structure ω is associated to a metric g on M . Recall that the space of Riemannian metrics on a manifold is connected. Therefore, the Morse trick may help to prove that the associated loop spaces are all symplectomorphic. Recall that the Moser’s trick for exact symplectic forms works as follows. Assume that the 1-parametric family of forms $\omega_t = d\mu_t$ are symplectic on a manifold N . We want to find $\phi_t : N \rightarrow N$, such that $\phi_t^* \omega_t = \omega_0$. Let Y_t be the vector field generating ϕ_t . Then

$$L_{Y_t} \omega_t = d\left(\frac{d\mu_t}{dt}\right)$$

is satisfied. This is true if

$$(10) \quad i_{Y_t} \omega_t = \frac{d\mu}{dt},$$

which has a unique solution since ω_t non-degenerate. In our case we are given two different Riemannian metrics g_0 and g_1 in M , so there is a path

of metrics g_t that joins them. Therefore there is a path of exact symplectic forms ω_t in $\mathcal{L}_p(M)$. Now we substitute into equation (10) in our case to obtain

$$(11) \quad \int_0^1 g_t \left(\frac{\nabla Y_t(\gamma)(s)}{ds}, X(s) \right) ds = \frac{d}{dt} \int_0^1 g_t(X(s), \gamma'(s)) ds,$$

for all $X \in \Gamma(S^1, \gamma^*TM)$. (Note that $Y_t(\gamma) \in T_\gamma \mathcal{L}_p(M)$.) This equation does not have continuous solutions in general. This is because we can compute $\frac{\nabla Y_t(\gamma)(s)}{ds}$ in a continuous way and we may assume that $Y_t(\gamma)(0) = 0$, but then there is no reason to expect that $Y_t(\gamma)(1) = 0$. So, it turns out that we do not get a uniqueness result for the symplectic structure. We will see a way of partially avoiding this obstruction. For this we are forced to change our point of view and work over the space $\mathcal{L}(M)$, where the forms ω are quasi-symplectic.

Proof of Theorem 1.2.

We try to apply Moser's trick in the space $\mathcal{L}(M)$. Recall that ω is not symplectic in this case, since it has a finite dimensional kernel. Denote by $\mathcal{G}(M)$ the space of Riemannian metrics over M .

Given a point $p \in M$, define

$$\mathfrak{so}_0(p) = \{a \in \mathfrak{so}(T_p M) ; \det(a) = 0\}.$$

Recall that $\mathfrak{so}_0(p) \neq \mathfrak{so}(T_p M)$ if $\dim(M)$ is even. In that case it is a codimension 1 (singular) submanifold. This defines a fibration

$$\mathfrak{so}_0(M) \rightarrow M,$$

which is a (non-linear) subbundle of the bundle $\mathfrak{so}(TM)$.

For a metric $g \in \mathcal{G}(M)$, the curvature R_g associated to the metric is a section of the bundle $\mathfrak{so}(TM) \otimes \Omega^2(M)$, therefore it defines a bundle map

$$R_g : \bigwedge^2(TM) \rightarrow \mathfrak{so}(TM).$$

Note that these bundles have the same rank. We say that a metric g is R-generic if for each $p \in M$ there are two vectors $u, v \in T_p M$ such that $\det(R_g(u, v)) \neq 0$, in the case $\dim M \geq 4$, or if R_g vanishes only at a discrete set of points, in the case $\dim M = 2$.

Define

$$\mathcal{G}_0(M) = \{g \in \mathcal{G}(M) : g \text{ is R-generic}\}.$$

Given two metrics g'_0 and g'_1 , there are two metrics g_0 and g_1 such that g_i is as close as desired to g'_i and g_i is R-generic. Moreover we may do the same for paths g_t of R-generic metrics. This is clear in the case $\dim M = 2$. If $n = \dim M \geq 4$, we work as follows: for each $g \in \mathcal{G}$ and $p \in M$, there exists a small perturbation of g which is R-generic in a neighborhood of p . This is true since the subspace $\text{Hom}(\bigwedge^2(T_p M), \mathfrak{so}_0(p)) \subset \text{Hom}(\bigwedge^2(T_p M), \mathfrak{so}(T_p M))$

is of codimension $\binom{n}{2}$, which is bigger than n . The result follows from a Sard-Smale lemma applied to the functional

$$\begin{aligned} \mathcal{G}(M) \times M &\rightarrow \text{Hom}\left(\bigwedge^2(TM), \mathfrak{so}(TM)\right) \\ (g, p) &\rightarrow R_g(p). \end{aligned}$$

Now define the set

$$\text{SO}_0(T_p M) = \{A \in \text{SO}(T_p M) ; A - \text{Id} \text{ is not invertible}\}.$$

This is a codimension 1 stratified submanifold of $\text{SO}(T_p M)$ and defines a bundle

$$\text{SO}_0(TM) \rightarrow M.$$

Let us define

$$\mathcal{L}_s(M) = \{\gamma \in \mathcal{L}(M) ; P_0^1 - \text{Id} \text{ is not invertible}\}.$$

There is a map

$$\begin{aligned} \theta : \mathcal{L}(M) &\rightarrow \text{SO}(TM) \\ \gamma &\rightarrow (P_0^1)_\gamma \in \text{SO}(T_{\gamma(0)} M). \end{aligned}$$

Clearly $\mathcal{L}_s(M) = \theta^{-1}(\text{SO}_0(TM))$. Therefore if θ is generic in a suitable sense, $\mathcal{L}_s(M)$ will be a stratified codimension 1 submanifold. We claim that if $g \in \mathcal{G}_0(M)$, this is the case. To check this, pick $\gamma \in \mathcal{L}_s(M)$. Being g an R-generic metric at $T_{\gamma(0)} M$, there exist two vectors $u, v \in T_{\gamma(0)} M$ such that $\det(R_g(u, v)) \neq 0$.

Recall [Be02, Subsection 15.4.1] that if we extend u, v to a neighborhood of $\gamma(0)$ in such a way that they define a local pair of coordinates (x, y) where

$$\begin{aligned} \frac{\partial}{\partial x} \Big|_{(0,0)} &= u, \\ \frac{\partial}{\partial y} \Big|_{(0,0)} &= v, \end{aligned}$$

and we define the path $\gamma_{u,v}^s$ as (in the coordinates (x, y)):

- $\gamma_{u,v}^s(t) = (4st, 0)$, $t \in [0, 1/4]$,
- $\gamma_{u,v}^s(t) = (s, 4s(t - 1/4))$, $t \in [1/4, 1/2]$,
- $\gamma_{u,v}^s(t) = (4s(3/4 - t), s)$, $t \in [1/2, 3/4]$,
- $\gamma_{u,v}^s(t) = (0, 4s(1 - t))$, $t \in [3/4, 1]$,

we obtain

$$\lim_{s \rightarrow 0} \frac{(P_0^1)_{\gamma_{u,v}^s}}{s} = R_g(u, v).$$

Take the path which is the juxtaposition of γ with $\gamma_{u,v}^s$,

$$\beta_s = \gamma * \gamma_{u,v}^s.$$

This family of paths determines a tangent vector in $T_\gamma \mathcal{L}(M)$. We will show that it is transverse to the submanifold $\mathcal{L}_s(M)$. The holonomy of β_s is

$$(12) \quad (P_0^1)_{\beta_s} = P_0^1 (\text{Id} + s R_g(u, v)) + O(s^2),$$

where P_0^1 is the holonomy of γ . Now embed $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ by complexifying. Then $P_0^1 \in \mathfrak{gl}(n, \mathbb{C})$ admits a Jordan canonical form

$$J = BP_0^1 B^{-1},$$

where $B \in \mathrm{GL}(n, \mathbb{C})$. Multiply on the left and on the right by B and B^{-1} the expression (12) to obtain

$$(13) \quad B(P_0^1)_{\beta_s} B^{-1} = J + sBP_0^1 R_g(u, v)B^{-1} + O(s^2).$$

Now, it is easy to check that the eigenvalues of the right hand side of (13), for s small enough, are far away from zero or grow faster than ϵs , for some fixed $\epsilon > 0$. Since B is just a change of coordinates matrix for $(P_0^1)_{\beta_s}$ on the left hand side of (13), the eigenvalues of $(P_0^1)_{\beta_s}$ are the same than those of the right hand side of (13). This implies that the tangent vector determined by β_s is transverse to $\mathcal{L}_s(M)$.

In the case $\dim M = 2$, we work as before when $\gamma(0)$ does not coincide with either of the points $p \in M$ with $R_g(p) = 0$. We can define $\mathcal{L}_s(M)$ to be the same set as before together with the paths starting at a point p with $R_g(p) = 0$. This latter set is of codimension n , so $\mathcal{L}_s(M)$ is still of codimension 1, as needed, in this case.

Recall that it is fundamental for this argument to work that the dimension of M is even. If the dimension is odd, then $\mathcal{L}_s(M) = \mathcal{L}(M)$.

All the computations done at the beginning of this subsection remain valid for the family $\{g_t\}$ and so we can follow Moser's trick to end up with equation (10) again. Now we recall that the solution Y_t is not unique, basically because we are working with $\mathcal{L}(M)$, and there is a space of parallel vector fields along γ , which are in the kernel of the quasi-symplectic form ω . In general we obtain $Y_t^w = Y_t + w$ as a valid solution where Y_t is a particular solution and w is a parallel vector field along γ (here we may have $w(0) \neq w(1)$ – actually, this will be the case). Again in general $Y_t^w(1) \neq Y_t^w(0)$. However a careful choice of w may help. We fix the equation

$$Y_t(0) + w(0) = Y_t(1) + w(1),$$

that if solved for some $w(0) \in \mathbb{R}^n$, leads to a smooth solution of equation (10). The previous equation leads to

$$(14) \quad P_0^1(w(0)) - w(0) = Y_t(1) - Y_t(0),$$

that clearly has a unique solution whenever $\gamma \notin \mathcal{L}_s(M)$. So equation (10) has a unique continuous solution outside a set of positive codimension in $\mathcal{L}(M)$. We are aiming to construct an “approximate” solution to (14). To get this, we can perturb equation (14) to

$$(15) \quad (\lambda P_0^1 - \mathrm{Id})(w_\lambda(0)) = Y_t(1) - Y_t(0),$$

which always admits a solution for $|\lambda| < 1$ since $P_0^1 \in \mathrm{SO}(n)$. Now we assume that λ is a smooth map $\lambda : \mathcal{L}(M) \times [0, 1] \rightarrow [1 - \epsilon, 1]$ satisfying

- (i) $\lambda(\gamma, t) = 1 - \epsilon$ if $\gamma \in \mathcal{L}_s(M)$ for the metric g_t .

(ii) $\lambda(\gamma, t) = 1$ on a small neighborhood of $\mathcal{L}_s(M)$.

This defines a family of flows $\{Y_t^\lambda\}$ (for the given constant $\epsilon > 0$). The integrals at time 1 of these flows are generating the family ϕ_ϵ required in the statement of the theorem.

The flow exists and is unique. This is due to the fact that it can be understood as a parametric (smoothly dependent) family of flows in M and there we have existence and uniqueness (for all times). The smooth dependency in the parameters gives that the flow is Fréchet smooth. We also need to prove that the flow is by diffeomorphisms since, being $\mathcal{C}^\infty(S^1, M)$ a Fréchet manifold, this is not automatic. But this follows from $\phi_{-\epsilon} \circ \phi_\epsilon = \text{Id}$, and so the flow maps admit inverses.

It is a routine to check that ϕ_ϵ takes the quasi-symplectic form associated to g_0 to a form ϵ -close in L^2 -norm to the quasi-symplectic form associated to g_1 . Moreover, since g_i and g'_i are as close as needed we can also claim that their associated quasi-symplectic structures are close to each other in L^2 -norm. \square

Finally, it is remarkable to note that the possibility of addition of parallel vector fields has been the key to find a continuously varying family of functions which are solutions to (10) almost everywhere. This is the geometric reason for which we cannot extend the computation to the case of the (symplectic) space $\mathcal{L}_p(M)$.

3. LOOP SPACES AS CONTACT MANIFOLDS.

We want to check whether the symplectic manifold $\mathcal{L}_p(M)$ has hypersurfaces of contact type on it. We prove now Theorem 1.3.

Proof of Theorem 1.3.

Let X be a vector field on M satisfying $L_X g = g$. Then $\nabla X \in \text{End}(TM)$, and its symmetrization is $\frac{1}{2} \text{Id}$. This follows since, for Y, Z vector fields on M , we have

$$\begin{aligned} g(\nabla_Z X, Y) + g(\nabla_Y X, Z) &= \\ &= g(\nabla_X Z, Y) + g(\nabla_X Y, Z) - g(L_X Z, Y) - g(L_X Y, Z) = \\ &= X(g(Y, Z)) + (L_X g)(Y, Z) - L_X(g(Y, Z)) = \\ &= g(Y, Z), \end{aligned}$$

where we have used that $L_X Z = \nabla_X Z - \nabla_Z X$ on the second line. The anti-symmetrization of ∇X is $\mathcal{A}(\nabla X) = \mathcal{A}(\nabla X^\#)_\# = (dX^\#)_\#$, where $X^\#$ is the 1-form associated to X ("raising the index"), and the $(\cdot)_\#$ means "lowering the index" with the metric. Recall that a vector field is "locally gradient-like" in a neighborhood U for a metric g if it is g -dual of some exact 1-form df , where f is a function $f : U \rightarrow \mathbb{R}$. Thus, if X is locally gradient-like, then $X^\#$ is a locally exact, i.e. closed, 1-form and so $\mathcal{A}(\nabla X) = 0$. Then $\nabla X = \frac{1}{2} \text{Id}$.

Associated to X there is an induced vector field \hat{X} on $\mathcal{L}(M)$. It is defined as follows: for $\gamma \in \mathcal{L}(M)$, $\hat{X}_\gamma \in T_\gamma \mathcal{L}(M)$ is given by $\hat{X}_\gamma(t) = X(\gamma(t))$. We want to check that $L_{\hat{X}}\mu = \mu$. For $Y \in T_\gamma \mathcal{L}(M)$, we have

$$\begin{aligned} \alpha(Y) &= i_{\hat{X}}\omega(Y) = \omega(\hat{X}, Y) = \\ &= \int_0^1 g\left(\frac{\partial X}{\partial t}, Y\right) dt = \int_0^1 g(\nabla_{\gamma'} X, Y) dt = \\ &= \frac{1}{2} \int_0^1 g(\gamma', Y) dt = \mu(Y). \end{aligned}$$

So $\alpha = \mu$. Then

$$L_{\hat{X}}\mu = di_{\hat{X}}\mu + i_{\hat{X}}d\mu = di_{\hat{X}}i_{\hat{X}}\mu + i_{\hat{X}}\omega = 0 + \alpha = \mu.$$

From this, it follows that $L_{\hat{X}}\omega = L_{\hat{X}}d\mu = dL_{\hat{X}}\mu = d\mu = \omega$, as required. \square

Remark 3.1. *The manifolds to which the previous result applies, that is, those satisfying $L_X g = g$ with X locally gradient-like, are locally of the form $(N \times \mathbb{R}, e^t(g + dt^2))$ with expanding vector field $X = \frac{\partial}{\partial t}$. This follows by writing $X = \text{grad } f$, with $f > 0$ and putting $t = \log(f)$.*

Two examples are relevant:

- $M = N \times \mathbb{R}$, with (N, g) a compact Riemannian manifold. Give M the metric $e^t(g + dt^2)$.
- Let (N, g) be an open Riemannian manifold with a diffeomorphism $\varphi : N \rightarrow N$ such that $\varphi^*(g) = e^\lambda g$, $\lambda > 0$. Then take $M = (N \times [0, \lambda]) / \sim$, where $(x, 0) \sim (\varphi(x), \lambda)$ and M has the metric induced by $e^t(g + dt^2)$.

To finish, let us check that the familiar finite dimensional picture translates to this case.

Proposition 3.2. *Let (M, g) be a Riemannian manifold which has a locally gradient-like vector field X satisfying $L_X g = g$. Then the hypersurface*

$$\mathcal{L}_{p,1}(M) = \{\gamma \in \mathcal{L}_p(M) ; \text{length}(\gamma) = 1\}$$

is a contact hypersurface of $\mathcal{L}_p(M)$.

Proof.

We need to check that $\alpha = i_{\hat{X}}\omega$ is a contact form on $\mathcal{L}_{p,1}(M)$. First we claim that α is nowhere zero on that submanifold. If this were not the case, then we would have that

$$(16) \quad (i_{\hat{X}}\omega)|_{\mathcal{L}_{p,1}(M)} = 0,$$

and we know that $i_{\hat{X}}\omega(\hat{X}) = 0$. Note that \hat{X} is transversal to $\mathcal{L}_{p,1}(M)$ since the flow increases the length of the loop. Therefore we have that $i_{\hat{X}}\omega = 0$ and ω would not be symplectic, which is a contradiction. So α is nowhere zero.

Now we have the distribution $(\ker \alpha, \omega = d\alpha)$ on $\mathcal{L}_{p,1}(M)$. To finish we check that it is symplectic. Assume that $Y \in T_\gamma \mathcal{L}_{p,1}(M)$ satisfies that

$$\omega(Y, Z) = 0,$$

for all $Z \in T_\gamma \mathcal{L}_{p,1}(M)$. Moreover we have that $\omega(Y, \hat{X}) = -\alpha(Y) = 0$. Hence $i_Y \omega = 0$, and we get a contradiction.

Corollary 3.3. *Given a Riemannian manifold (M, g) , the manifold $(M \times \mathbb{R}, e^\lambda(g + d\lambda^2))$ has an associated space of loops of length one with a canonical contact form. For a loop γ and Y vector field along γ , we denote $\gamma = (\gamma_1, \gamma_2)$ and $Y = (Y_1, Y_2)$ according to the decomposition $M \times \mathbb{R}$. Then the contact form is given by*

$$\alpha(Y) = \mu(Y) = \frac{1}{2} \int_0^1 e^{\gamma_2(t)} (g(\gamma'_1(t), Y_1(t)) + \gamma'_2(t) Y_2(t)) dt.$$

3.1. Reeb vector fields. We compute the Reeb vector field associated to the contact form. We need to do it in $\mathcal{L}_1(M)$, that is a quasi-contact space (instead of contact). This is necessary in order to obtain a smooth Reeb vector field.

Lemma 3.4. *The Reeb vector field associated to $(\mathcal{L}_1(M), \alpha)$ is the vector*

$$R = \frac{\gamma'}{\|\gamma'\|} \cdot h(\gamma),$$

where $h : \mathcal{L}_1(M) \rightarrow \mathbb{R}$ is a $\text{Diff}(S^1)$ -invariant strictly positive function on the loop space.

Proof.

The condition for a vector $V \in T_\gamma \mathcal{L}(M)$ to belong to $T_\gamma \mathcal{L}_1(M)$ is

$$\lim_{s \rightarrow 0} \frac{\int_0^1 \|\gamma' + s \frac{dV}{dt}\| dt}{s} = 0.$$

So we are asking the vector to satisfy

$$\int_0^1 g\left(\frac{\partial V}{\partial t}, \gamma'(t)\right) \frac{1}{\|\gamma'\|} dt = 0.$$

Thus the previous equation can be rewritten as

$$\omega\left(V, \frac{\gamma'}{\|\gamma'\|}\right) = 0.$$

Imposing this condition for every non-zero vector $V \in T_\gamma \mathcal{L}_1(M)$, we get that the Reeb vector field is a positive multiple of $\frac{\gamma'}{\|\gamma'\|}$, so proving the statement. It remains to be checked that $h(\gamma)$ is invariant by changes of parametrization preserving the origin. This is a consequence of the invariance under changes of coordinates of the defining equation

$$\alpha(R) = i_{\hat{X}} \omega\left(\frac{\gamma'}{\|\gamma'\|} h(\gamma)\right) = 1.$$

□

There are more solutions to the Reeb vector field equation since for any parallel vector field w along γ , we can add it to R , so that $R + w$ defines another solution to the equation. However those solutions are not continuous (as functionals on $\mathcal{L}(M)$) in general. This is the reason that prevents us to define the Reeb field in the space $\mathcal{L}_{p,1}(M)$. Again the flexibility of the quasi-contact structure is the key to find the solution.

We have the following

Lemma 3.5. *All the Reeb orbits of α are closed. For a loop*

$$\begin{aligned}\gamma : S^1 &\rightarrow M \\ t &\rightarrow \gamma(t),\end{aligned}$$

the Reeb orbit passing through it has period equal to the length of γ divided by $h(\gamma)$.

Proof.

Take $\gamma : S^1 \rightarrow M$. The arc-length parametrization of $\gamma(S^1)$ is denoted as γ_p , where p is the point of $\gamma(S^1)$ in which the arc-length parameter starts. So we define $\theta(s, t) = \gamma_{\gamma(t)}(h(\gamma)s)$ which is clearly the Reeb orbit starting at γ . It is periodic with period length $(\gamma)/h(\gamma)$. □

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